W58. In all triangle ABC holds

$$\sum \sqrt{\frac{a(h_a - 2r)}{(3a + b + c)(h_a + 2r)}} \le \frac{3}{4}.$$

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Let s be semiperimeter of $\triangle ABC$. Since $h_a = \frac{bc}{2R}$ and $abc = 4Rrs$

then $\frac{h_a - 2r}{h_a + 2r} = \frac{\frac{bc}{2R} - 2r}{\frac{bc}{2R} + 2r} = \frac{bc - 4Rr}{bc + 4Rr} = \frac{sbc - 4Rrs}{sbc + 4Rrs} = \frac{sbc - abc}{sbc + abc} =$

$\frac{s - a}{s + a} = \frac{b + c - a}{3a + b + c}$ and, therefore,

$$\sum \sqrt{\frac{a(h_a - 2r)}{(3a + b + c)(h_a + 2r)}} = \sum \frac{\sqrt{a(b + c - a)}}{3a + b + c}.$$

Noting that $\sqrt{a(b + c - a)} = \frac{1}{2}\sqrt{4a(b + c - a)} \le \frac{1}{2} \cdot \frac{4a + (b + c - a)}{2} =$

$\frac{3a + b + c}{4}$ we obtain $\sum \frac{\sqrt{a(b + c - a)}}{3a + b + c} \le \frac{1}{4} \sum \frac{3a + b + c}{3a + b + c} = \frac{3}{4}$.

Since in inequality $\sqrt{a(b + c - a)} \le \frac{3a + b + c}{4}$ hold iff $4a = b + c - a \Leftrightarrow$

$b + c = 5a$ then equality in original inequality can be attained iff

(**1**) $\begin{cases} b + c = 5a \\ c + a = 5b \\ a + b = 5c \end{cases}$. But it is impossible because $a, b, c > 0$ and (**1**)

implies $a + b + c = 0$.

Thus, $\frac{3}{4}$ isn't attainable upper bond for $\sum \sqrt{\frac{a(h_a - 2r)}{(3a + b + c)(h_a + 2r)}}$.